# Application of type B and Monte Carlo methods for UA: examples for discharge and volume measurements in a circular sewer pipe

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## 1. INTRODUCTION

In this example, the uncertainty in the discharge and volume measured in a circular sewer pipe is calculated by means of two methods: i) the GUM type B method by application of the law of propagation of uncertainties LPU (ISO, 2009a) and ii) the Monte Carlo method (ISO, 2008, 2009b). For pedagogical reasons, most algebra and calculations are presented in detail.

## 2. DISCHARGE MEASUREMENT

Let consider a circular concrete sewer pipe with radius $R = 0.5$. It is assumed i) that the pipe is circular and not affected by any deformation, and ii) that there are no deposits on the invert.

The discharge $Q$ ($m^3/s$) is then given by

$$Q(R,h,U) = S(h)U = R^2\left[\text{Arccos}\left(1-\frac{h}{R}\right)-\left(1-\frac{h}{R}\right)\sqrt{1-\left(1-\frac{h}{R}\right)^2}\right]U \qquad \text{eq. 1}$$

where $h$ (m) is the water level and $U$ (m/s) the mean flow velocity.

Calculations have been made with the following values:

$R = 0.5$ m, $u(R) = 0.001$ m

$h = 0.7$ m, $u(h) = 0.005$ m

$U = 0.8$ m/s, $u(U) = 0.05$ m/s.

Appendix 5.1 explains how the standard uncertainties $u(R)$, $u(h)$ and $u(U)$ have been estimated.

The resulting discharge $Q = 0.4697$ $m^3/s$. Note here that all results in the paper will be given with 4 or more digits only for illustration and comparison purposes. Under real conditions of application, one, two or three digits would be sufficient: the additional ones appear in *italic characters* in numerical values. However, it is of course recommended to keep the maximum number of digits in all intermediate calculations.

### 2.1 TYPE B ESTIMATION

All measured variables $R$, $h$ and $U$ are measured independently with different instruments and are not correlated. Consequently, the law of propagation of uncertainty (LPU) can be written

$$u(Q)^2 = u(R)^2\left(\frac{\partial Q}{\partial R}\right)^2 + u(h)^2\left(\frac{\partial Q}{\partial h}\right)^2 + u(U)^2\left(\frac{\partial Q}{\partial U}\right)^2 \qquad \text{eq. 2}$$

The partial derivatives and their numerical values are equal to

$$\left(\frac{\partial Q}{\partial R}\right) = 2UR\,\text{Arccos}\left(1-\frac{h}{R}\right) - 2U\sqrt{2hR-h^2} = 0.852\,638\,427\,096 \text{ m}^2/\text{s} \qquad \text{eq. 3}$$

$$\left(\frac{\partial Q}{\partial h}\right) = 2U\sqrt{2hR-h^2} = 0.733\,212\,111\,192 \text{ m}^2/\text{s} \qquad \text{eq. 4}$$

$$\left(\frac{\partial Q}{\partial U}\right) = R^2 \text{ Arccos}\left(1-\frac{h}{R}\right) - (R-h)\sqrt{2hR-h^2} = 0.587\ 229\ 807\ 114 \text{ m}^2 \qquad \text{eq. 5}$$

In case algebra is considered too difficult, the above exact analytical expressions can be replaced by numerical estimations of partial derivatives by applying a second order approximation operator (several digits are given only for comparison between exact and approximated values):

$$\left(\frac{\partial Q}{\partial R}\right) = \frac{Q(R+\delta_R,h,U)-Q(R-\delta_R,h,U)}{2\delta_R} \approx \frac{0.469792 - 0.469775}{2.10^{-5}} \approx 0.852\,638\,427\,287 \text{ m}^2/\text{s} \qquad \text{eq. 6}$$

with $\delta_R = 10^{-5}$ m

$$\left(\frac{\partial Q}{\partial h}\right) = \frac{Q(R,h+\delta_h,U)-Q(R,h-\delta_h,U)}{2\delta_h} \approx \frac{0.469791 - 0.469776}{2.10^{-5}} \approx 0.733\,212\ \ 111\ \ 117 \text{ m}^2/\text{s} \qquad \text{eq. 7}$$

with $\delta_h = 10^{-5}$ m

$$\left(\frac{\partial Q}{\partial U}\right) = \frac{Q(R,h,U+\delta_U)-Q(R,h,U-\delta_U)}{2\delta_U} \approx \frac{0.469789 - 0.469777}{2.10^{-5}} \approx 0.587\,229\,807\,111 \text{ m}^2 \qquad \text{eq. 8}$$

with $\delta_U = 10^{-5}$ m/s

It is important to note that $\delta_x$ values should be chosen in such a way that $\delta_x \ll u(x)$.

Eq. 2 gives

$$u(Q)^2 = 7.2699.10^{-7} + 1.3440.10^{-5} + 8.6209.10^{-4} = 8.7626.10^{-4} \text{ m}^6/\text{s}^2 \qquad \text{eq. 9}$$

and then the standard uncertainty $u(Q) = 0.0296$ m$^3$/s ≈ 0.03 m$^3$/s.

With the enlargement factor $k_e = 2$, $Q \pm k_e.u(Q) = 0.4697 \pm 0.0592$ m$^3$/s.

i.e. a relative enlarged uncertainty $\frac{k_e u(Q)}{Q} = \frac{0.0593}{0.4697} = 0.126 = 12.6$ %.

This can be interpreted as the true value $Q$ has an approximately 95 % probability to lie in the interval $[Q - k_e u(Q), Q + k_e u(Q)] = [0.4106, 0.5289] \approx [0.41, 0.53]$.

Under real conditions, one would use $Q = 0.47 \pm 0.06$ m$^3$/s.

One should note that, in eq. 9, the first term ($7.26.10^{-7}$) is negligible compared to the two other ones: the contribution of the uncertainty in $R$ to the total uncertainty in $Q$ can be ignored. In addition, the contribution of the uncertainty in $h$ is lower than the contribution of the uncertainty in $U$. However, this conclusion is not valid for all possible values of $R$, $h$ and $U$: a specific analysis can be made for each particular set of values ($R$, $h$, $U$).

## 2.2 MONTE CARLO ESTIMATION

For this example, $N = 10^6$ simulations are run. Details of calculation are given in Appendix 5.2.

The following samples are created:

$N$ values of $R$ normally distributed with mean value $\overline{R} = 0.5$ m and standard deviation $s(R) = 0.001$ m.

$N$ values of $h$ normally distributed with mean value $\overline{h} = 0.7$ m and standard deviation $s(h) = 0.005$ m.

$N$ values of $U$ normally distributed with mean value $\overline{U} = 0.8$ m/s and standard deviation $s(U) = 0.05$ m/s.

All samples are independent and not correlated. The histogram of the water level sample is shown in Figure 1. The coefficients of correlation between the three samples are given in Table 1: the samples are clearly not correlated.

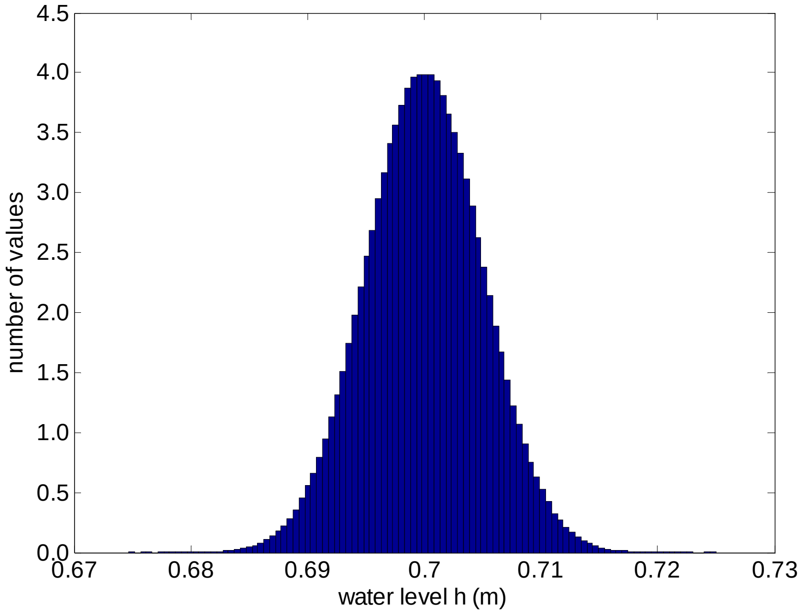


Figure 1 : histogram of the water level sample (one million values of *h*).

| | *R* | *h* | *U* |
|---|---|---|---|
| *R* | 1 | -0.0004 | -0.0009 |
| *h* | | 1 | +0.0005 |
| *U* | | | 1 |

Table 1 : coefficients of correlation between the samples of the three variables *R* (m), *h* (m) and *U* (m/s).

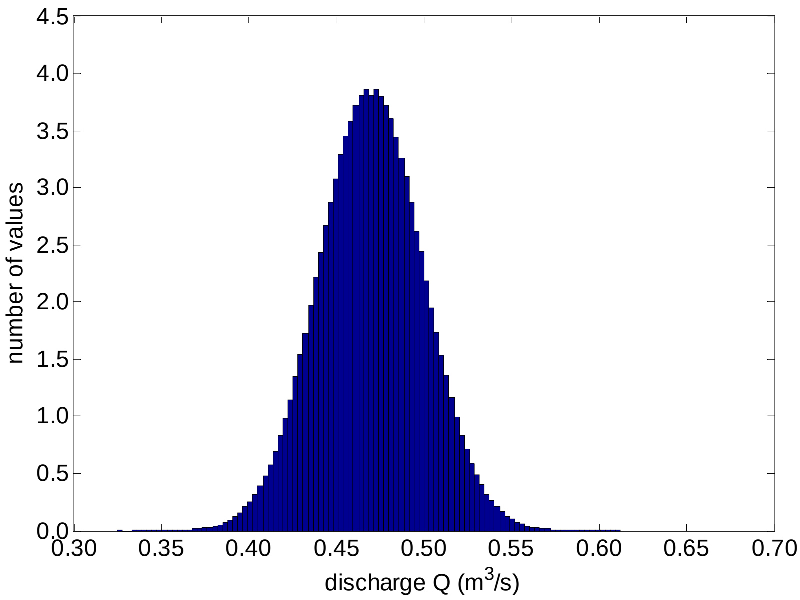


Figure 2 : histogram of the discharge sample (one million values of *Q*).

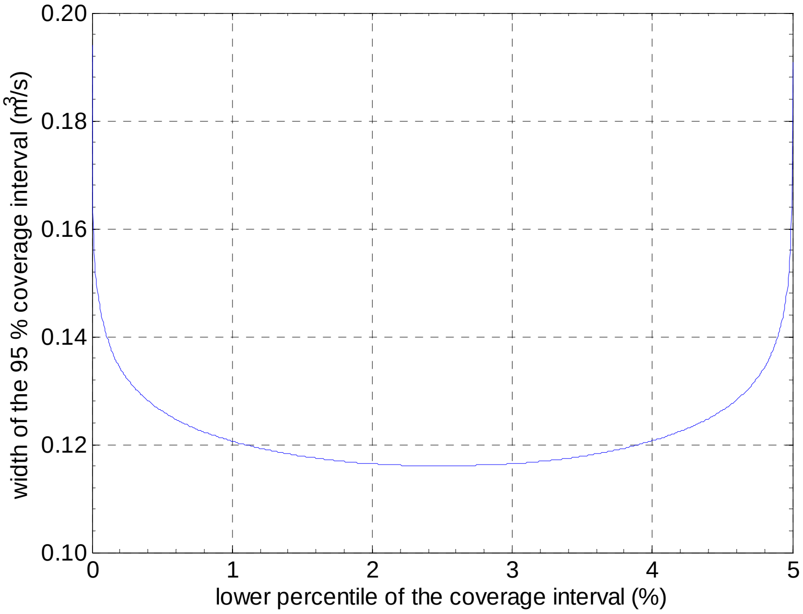


Figure 3 : width of the 95% coverage interval vs. the lower percentile of the coverage interval.

The histogram of the resulting $N$ values of discharge $Q$ is shown in Figure 2. The mean value $\overline{Q}$ is equal to 0.46*98* $m^3/s$.

The variation of the width of the 95% coverage interval is represented in Figure 3. The shortest 95% coverage interval is [0.41*19*, 0.52*79*] ≈ [0.41, 0.53].

Simulations with other samples of the same size $N = 10^6$ give slightly different values. This is illustrated in Table 2 with 10 uncertainty evaluations. The standard deviation in the bottom line reveals that the dispersion of the results is very small and, in this case, negligible.

| UA # | $\overline{Q}$ ($m^3/s$) | shortest 95 % CI | |
|---|---|---|---|
| 1 | 0.4698 | 0.4115 | 0.5275 |
| 2 | 0.4698 | 0.4112 | 0.5273 |
| 3 | 0.4698 | 0.4118 | 0.5277 |
| 4 | 0.4698 | 0.4122 | 0.5283 |
| 5 | 0.4698 | 0.4118 | 0.5279 |
| 6 | 0.4698 | 0.4116 | 0.5275 |
| 7 | 0.4698 | 0.4118 | 0.5278 |
| 8 | 0.4698 | 0.4120 | 0.5278 |
| 9 | 0.4697 | 0.4114 | 0.5276 |
| 10 | 0.4698 | 0.4125 | 0.5284 |
| Mean value | 0.46*98* | 0.41*18* | 0.52*78* |
| Standard deviation | 0.0000 | 0.0004 | 0.0003 |

Table 2 : 10 Monte Carlo uncertainty assessments (UA) with $N = 10^6$.

## 2.3 COMPARISON

The type B method gives $\overline{Q}$ = 0.46*98* $m^3/s$ and the 95 % confidence interval is [0.41*06*, 0.52*89*] with $k_e = 2$.

The Monte Carlo method gives $\overline{Q}$ = 0.46*98* $m^3/s$ and the shortest 95 % coverage interval is [0.41*19*, 0.52*79*].

When considering only 2 digits as normally applied in practice, both results are considered similar.

If $k_e = 1.96$ is used instead of $k_e = 2$ (resp. a 95 % confidence interval instead of a 95.5 % confidence interval in case of a normal distribution), the type B interval becomes [0.41*17*, 0.52*78*] which is closer to the Monte Carlo shortest 95 % coverage interval.

Considering the Monte Carlo method as the reference method, one may conclude that, in this case, the type B method is validated and can be applied routinely, e.g. to discharge time series. The type B method requires preliminary algebra compared to the Monte Carlo method but, if it is validated, it runs faster than the Monte Carlo method when applied for example to time series. However, for single uncertainty assessment, the Monte Carlo method may be faster.

# 3. VOLUME MEASUREMENT

The second example deals with the estimation of the uncertainty in the volume calculated from discharge measurements. Let consider the volume $V(T)$ which is the integral of the discharge $Q(t)$ from time $T$-θ to time $T$:

$$V(T) = \int_{t=T-\theta}^{t=T} Q(t)dt \qquad \text{eq. 10}$$

In practice, $Q(t)$ is not measured continuously but with a discrete time step $\Delta t$. In addition, the true values of $Q(t)$ are not known: they are estimated by the discrete measured values $q(j\Delta t)$. The true volume $V(T)$ is then estimated approximately by the discrete sum $V_e(T)$:

$$V_e(T) = \sum_{j=1}^{n} q\big(T - (n-(j-1))\Delta t\big)\Delta t \qquad \text{eq. 11}$$

with $n$ the number of time steps $\Delta t$ for the duration $\theta = n\Delta t$ and $j$ the time index.

Assuming that the uncertainty in the value of $\Delta t$ is negligible, two main sources of uncertainty affect the estimate $V_e(T)$: i) uncertainties in the measured values $q(j\Delta t)$, which are estimated e.g. as shown in section 2 above, ii) uncertainties due to the discretisation of the true continuous signal $Q(t)$, which will be analysed hereafter. Uncertainties due to the discretisation have themselves two sources: i) the fact that the starting time of the measurement duration θ is arbitrarily, i.e. randomly, decided among all possible starting times uniformly distributed within the first time step $\Delta t$, ii) the fact that the exact integral of the continuous signal is replaced by a numerical sum of a limited number of discrete values.

Let re-write the discrete times as follows:

$$t_j = T + t_0 - (n - j + 1)\,\Delta t \qquad \text{eq. 12}$$

$t_0$ is the time of the first discrete measurement of the discharge. It is considered as a random variable within the first time step window $[T - n\Delta t, T - (n - 1)\Delta t]$. Once $t_0$ is chosen, all subsequent discrete times $t_j$ are determined by eq. 12.

The measured discharge $q(t_j)$ can be written:

$$q(t_j) = Q(t_j) + e(t_j) \qquad \text{eq. 13}$$

with $Q(t_j)$ the true value of the discharge $Q$ and $e$ the error (i.e. the difference between the true but unknown value $Q$ and the measured value $q$) at time $t_j$. The variance of $q(t_j)$ is equal to the variance of $e(t_j)$ as $Q(t_j)$ is the true value. Consequently,

$$u\big(q(t_j)\big)^2 = u\big(e(t_j)\big)^2 = \mathrm{var}\big(e(t_j)\big) \qquad \text{eq. 14}$$

eq. 13 and the algebra presented in section 3.1 hereafter are a simplified re-writing of a more detailed approach presented in Joannis and Bertrand-Krajewski (2009).

In order to illustrate the application of the above concepts and equations, true values of $Q(t)$ and $V(T)$ are necessary. That is the reason why a fictitious reference hydrograph $Q(t)$ has been created as the true value of a discharge time series (Figure 4). It corresponds to a 24 hour hydrograph at the outlet of a combined sewer system from 00:00 to 23:59 (1440 minutes) with a storm event occurring during the night between 04:00 and 06:00 (minutes 240 to 360). The hydrograph is the sum of constant and sinus continuous functions, allowing calculating the exact true value of the volume $V(T)$ in 24 hours, equal to 3000 $m^3$. The uncertainty in estimates $V_e(T)$ will be evaluated by both the type B approach and Monte Carlo simulations.

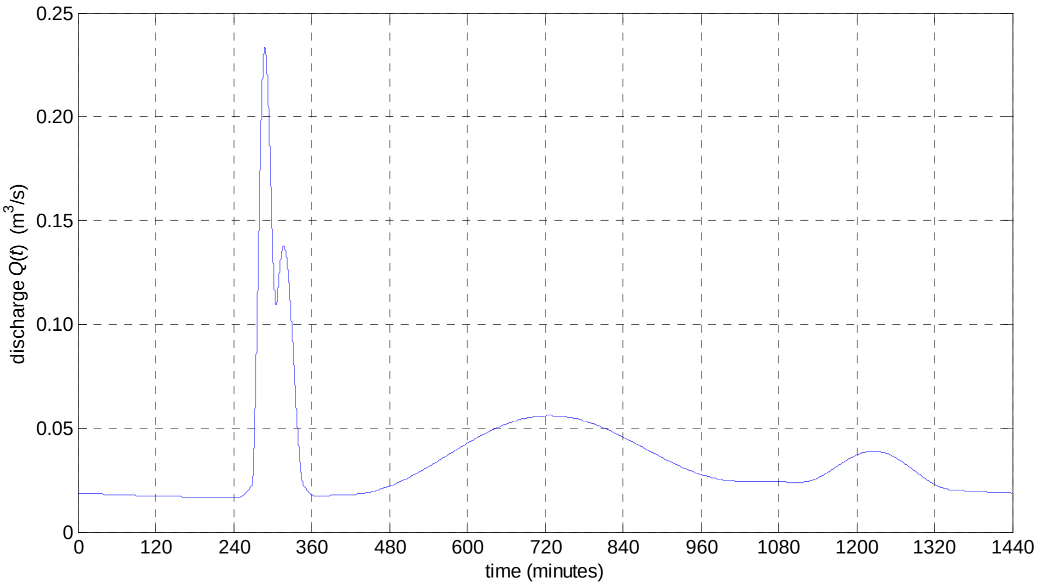


Figure 4 : fictitious 24 hour reference hydrograph.

### 3.1 TYPE B ESTIMATION

Let define

$$\Delta Q(t_j) = \left(\frac{V(T)}{n\Delta t}\right) - Q(t_j) \qquad \text{eq. 15}$$

$\Delta Q(t_j)$ is the distance between each true discharge value $Q(t_j)$ and the true mean discharge, in the interval [$T$-θ, $T$]. Each discharge value is considered here as an approximation of the mean value: this allows distributing the error in the estimate $V_e(T)$ equally for each time $t_j$. This hypothesis, combined with eq. 12 and eq. 13, leads to

$$V_e(T) = V(T) + \sum_{j=1}^{n}\left[\Delta Q(t_j) + e(t_j)\right]\Delta t \qquad \text{eq. 16}$$

With the hypothesis that $\Delta t$ has a negligible uncertainty, it is convenient to re-write the above equation as follows:

$$\frac{V_e(T)}{\Delta t} = \frac{V(T)}{\Delta t} + \sum_{j=1}^{n}\left[\Delta Q(t_j) + e(t_j)\right] \qquad \text{eq. 17}$$

Then, accounting for the fact that $V(T)$ is the true value of the volume and consequently has no uncertainty, the LPU applied by accounting for all covariances leads to

$$\begin{aligned}\frac{u(V_e(T))^2}{\Delta t^2} = & \sum_{j=1}^{n} u\left(e(t_j)\right)^2 + \sum_{j=1}^{n} u\left(\Delta Q(t_j)\right)^2 + 2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} u(\Delta Q(t_j),\Delta Q(t_k)) \\ & + 2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} u\left(e(t_j),e(t_k)\right) + 2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} u(\Delta Q(t_j),e(t_k))\end{aligned} \qquad \text{eq. 18}$$

According to eq. 14, $u(e(t_j) = u(q(t_j)$.

According to eq. 15, $u\left(\Delta Q(t_j)\right) = u\left(Q(t_j)\right)$ and $u\left(\Delta Q(t_j), \Delta Q(t_k)\right) = u\left(Q(t_j), Q(t_k)\right)$ as $V(T)$ is a true value. If $t_j$ and $t_k$ are known, $Q(t_j)$ and $Q(t_k)$ are also known and have no uncertainty. They can be considered as random variables only if $t_j$ and $t_k$ are random variables. If the time step $\Delta t$ is fixed, the possible variations of the $j$-th measurement $q(t_j)$ used in eq. 11 correspond to uniformly distributed variations of $t_j$ in the time step interval $[(T-n-j-1)\Delta t, (T-n-j)\Delta t]$. In practice, all values of $t_j$ are fixed if $t_0$ is given (eq. 12). Consequently, the possible variations of $q(t_j)$ are only due the variations of $t_0$, which is the key variable when considering uncertainties due to discretisation. $u(Q(t_j))^2$ and $u(Q(t_j),Q(t_k))$ are thus respectively the variance and covariance of the discharge values when $t_0$ is randomly chosen in a time interval of length equal to $\Delta t$. In practice, the continuous true discharge $Q$ is not known: only discrete measured values $q$ are available. A model is then necessary to interpolate between discrete values $q$ to re-build a virtual continuous signal. It is important to note that the covariance $u(Q(t_j),Q(t_k))$ is not the covariance between successive values $Q(t_j)$, $Q(t_j+\Delta t)$, $Q(t_j+2\Delta t)$, etc., but the covariance of the values of $Q(t_j)$ when $t_j$ is varying *within* the $j$-th time step when $t_0$ is chosen randomly to start a measurement period.

Considering lastly that i) there is no covariance between $\Delta Q$ and $e$ (measurement errors are not correlated with discretisation errors) and ii) there is no covariance between measurement errors (only random errors are considered, systematic errors have been corrected), eq. 18 can be simplified and re-written as follows:

$$u(V_e(T))^2 = \Delta t^2 \left( \underbrace{\sum_{j=1}^{n} u(q(t_j))^2}_{(A)} + \underbrace{\sum_{j=1}^{n} u(\Delta Q(t_j))^2}_{(B)} + \underbrace{2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} u(\Delta Q(t_j),\Delta Q(t_k))}_{(C)} \right) \quad \text{eq. 19}$$

To simulate a real measurement process, discrete hydrographs have been generated. An example is shown in Figure 5 with $\Delta t$ = 30 min and random errors $e(t_j)$ sampled from normal distributions with mean values equal to zero and standard deviations equal to 7.5 % of the true values $Q(t_j)$, i.e. $u(q(t_j)) = 0.075 \times Q(t_j)$. $\Delta t$ is large (30 min) only to facilitate the legibility of Figure 5. $V_e(T)$ is equal to 2967 m$^3$.

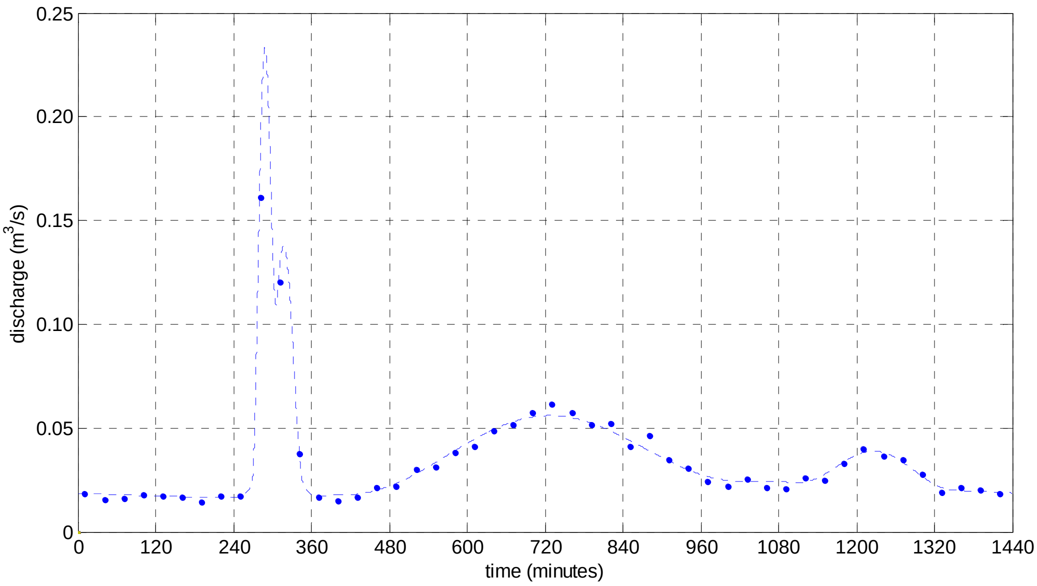


Figure 5 : example of discrete measurements with $\Delta t$ = 30 min and relative standard uncertainties in $q$ values equal to $0.075 \times q$. For comparison, the true signal is represented by the dashed line.

eq. 19 contains 3 terms:

- A corresponds to the random errors in measurements: A = 1541 m$^6$.

- B corresponds to the variances linked to the discretisation: B = 15409 m$^6$.

- C corresponds to the covariances linked to the discretisation: C = -7291 m$^6$.

The total uncertainty in $V_e(T)$ is then given by $u(V_e(T)) = \sqrt{A+B+C}$ = 98 m$^3$. With an enlargement factor $k_e = 2$, $V_e(T) \pm k_e\, u(V_e(T))$ = 2967 ± 196 m$^3$. The corresponding interval is [2771; 3163].

The contribution of the discretisation to the total uncertainty is equal to (B+C)/(A+B+C) = 84 % of the total uncertainty, while the contribution of random errors is only 16 %.

### 3.2 MONTE CARLO ESTIMATION

Monte Carlo simulations have been run in order to illustrate a more detailed analysis of the uncertainty in $V_e(T)$. The first step will consider only discretisation. The second step will consider both discretisation and measurement uncertainties as sources of uncertainty in $V_e(T)$.

#### 3.2.1 Effect of discretisation

This step aims to answer the following question: how discretisation of the true hydrograph $Q(t)$ with a time step $\Delta t$ may affect the estimate $V_e(T)$? Measured values $q(t_j)$ are considered with no uncertainties.

Fourteen time steps ($s$ = 1:14) have been analysed, respectively equal to 1, 2, 5, 10, 15, 20, 30, 45, 60, 90, 120, 180, 240 and 360 minutes. Long time steps have been tested only for illustration purposes as such high values are not realistic for discharge measurements. For each value $\Delta t_s$, the starting time $t_0$ is randomly set within the first time step [0, $\Delta t_s$]. One million ($N = 10^6$ ) simulations of $t_0$ have been generated by

$$t_{0si} = \alpha_i \Delta t_s \qquad \text{eq. 20}$$

with $\alpha_i$ a random number between 0 and 1 and $i$ the Monte Carlo simulation index $i$ = 1:$N$.

As a result, $N$ values of $V_e(T)$ are calculated for each time step $\Delta t_s$ :

$$V_e(T)_{si} = \left( \sum_{k=1}^{m_s} Q\left(t_{0si} + (k-1)\Delta t_s\right) \right) \Delta t_s \qquad \text{eq. 21}$$

with $m_s$ the number of measured discrete values for each time step $\Delta t_s$

$$m_s = \frac{1440}{\Delta t_s} \qquad \text{eq. 22}$$

Results are given in Figure 6. On each box plot, the central horizontal line indicates the median value, the central box is delimited by the first (bottom) and third (top) quartiles, and the extremities of the dashed lines represent the minimum (bottom) and maximum (top) values. Mean values, standard deviations and shortest 95 % coverage intervals are given in Table 3. All mean values are equal to 3000 m$^3$.

For all time steps from 1 to 20 minutes, the discretisation has a negligible effect: $V(T)$ is always precisely estimated by $V_e(T)$. For time steps ranging from 30 to 60 minutes, a significant dispersion of the values is observed. For example, for $\Delta t$ = 45 min, the mean is equal to 3000 m$^3$, the median is equal to 2976 m$^3$, but extreme values are respectively 2767 et 3273 m$^3$, i.e. -7.8 % and +9.1 % compared to the mean. In practice, this level of precision is acceptable. For time steps greater than 60 min, the dispersion increases dramatically and becomes more asymmetrical. For $\Delta t$ = 240 min, the median is equal to 2532 m$^3$ (-15.6 %) and extreme values are respectively 2399 m$^3$ (-20.0 %) and 5718 m$^3$ (+90.6 %).

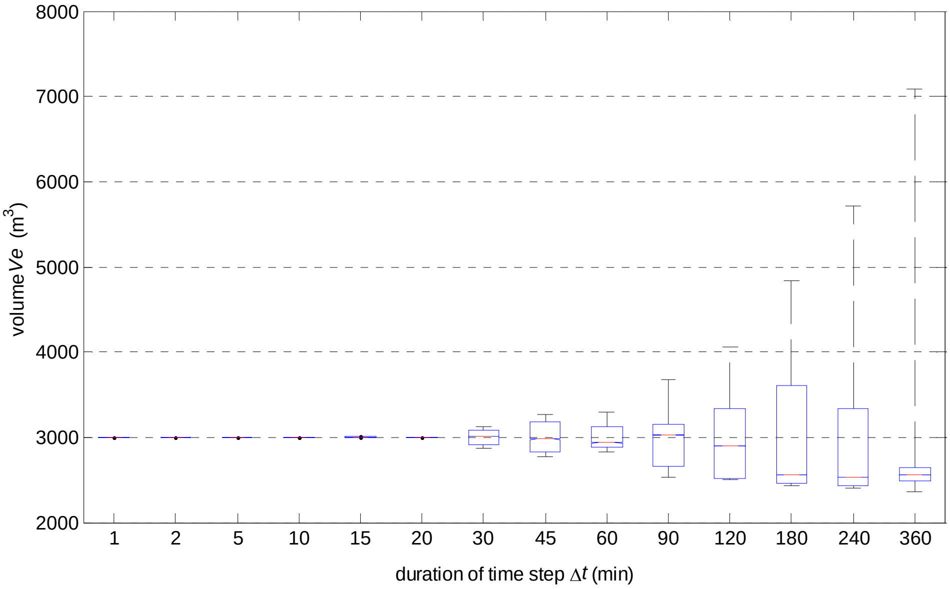


Figure 6 : effect of discretisation on the estimate $V_e(T)$.

### 3.2.2 Effect of discretisation and random errors

In this second step, the discrete measured values are no longer considered as true values but are affected by random errors $e(t_j)$ sampled from normal distributions with mean values equal to zero and standard deviations equal to 7.5 % of the true values $Q(t_j)$, i.e. $u(q(t_j)) = 0.075 \times Q(t_j)$, as in section 3.1. Consequently, one calculates

$$V_e(T)_{si} = \left( \sum_{k=1}^{m_s} Q\left(t_{0si} + (k-1)\Delta t_s\right) \times \left(1 + 0.075 \times e_{sik}\right) \right) \Delta t_s \qquad \text{eq. 23}$$

with $e_{sik}$ the random error for $Q_{sik} = Q\left(t_{0si} + (k-1)\Delta t_s\right)$.

Five millions ($N = 5 \times 10^6$) simulations have been run to ensure stable boundaries of the 95 % coverage intervals for all values of $\Delta t$. Results are shown in Figure 7. Mean values, standard deviations and shortest 95 % coverage intervals of $V_e(T)$ are given in Table 3.

Compared to the previous step (discretisation only), standard deviations slightly increase due to random errors in measured values. For time steps of less than 20 min, uncertainty due to discretisation is either negligible or very low and the uncertainty due to random errors is the most important contribution to the uncertainty in $V_e(T)$. However, the 95 % coverage intervals are very narrow (maximum ±2 % of the mean value). For time steps equal to or greater than 30 min, the discretisation is the most important source of uncertainty in $V_e(T)$. When the time step increases, $m_s$ decreases and the contribution of random errors in a decreasing number of measured values increases in absolute value. But this increasing contribution to the total uncertainty in $V_e(T)$ increases less rapidly, relatively, than the contribution due to the discretisation.

In Table 3 column E, the line corresponding to $\Delta t = 30$ min indicates that $u(V_e(T)) = 99$ m$^3$. This value is similar to the standard uncertainty of 98 m$^3$ calculated by means of the type B approach in section 3.1. The 95 % coverage interval in column F is [2827; 3170], to be compared with is [2771; 3163] obtained in section 3.1: both intervals can be considered equivalent (difference in lower and upper boundaries are resp. -2 % and -0.2 % with the interval in column F as the reference interval). However, it appears that 95 % confidence intervals calculated with the type B approach and the 95 % coverage intervals calculated with MC simulations will no longer be equivalent when $\Delta t$ increases, because of the asymmetrical distribution of $V_e(T)$. In this case, MC simulations are a better and less biased approach.

The MC method illustrated in this section for the calculation of a cumulated volume will be further developed for a more critical issue: the calculation of pollutant loads when discrete samples are taken in sewer systems for laboratory analyses. Discrete sampling strategies will be compared to on line continuous measurements (e.g. turbidity) collected with short time steps (e.g. 2 minutes): in this case, on line water quality time series will be considered as the true reference signal and uncertainties due to both discrete sampling and laboratory analyses will be assessed.

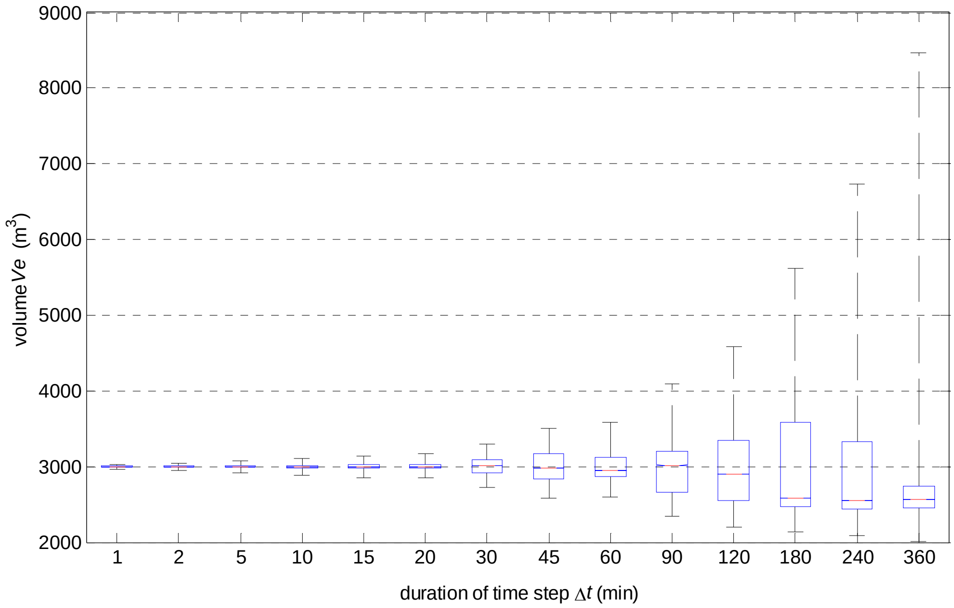


Figure 7 : effect of both discretisation and random errors on the estimate $Ve(T)$.

| $\Delta t$ (min) | discretisation only | | | discretisation + random errors | | |
|---|---|---|---|---|---|---|
| | (A) mean $V_e(T)$ ($m^3$) | (B) $u(V_e(T))$ ($m^3$) | (C) 95 % coverage interval | (D) mean $V_e(T)$ ($m^3$) | (E) $u(V_e(T))$ ($m^3$) | (F) 95 % coverage interval |
| 1 | 3000 | $< 10^{-10}$ | [3000; 3000] | 3000 | 7.6 | [2985; 3015] |
| 2 | 3000 | $< 10^{-10}$ | [3000; 3000] | 3000 | 10.7 | [2979; 3021] |
| 5 | 3000 | $< 10^{-10}$ | [3000; 3000] | 3000 | 17 | [2967; 3033] |
| 10 | 3000 | $< 10^{-10}$ | [3000; 3000] | 3000 | 24 | [2953; 3047] |
| 15 | 3000 | 6 | [2993; 3009] | 3000 | 30 | [2941; 3059] |
| 20 | 3000 | $< 10^{-10}$ | [3000; 3000] | 3000 | 34 | [2933; 3066] |
| 30 | 3000 | 90 | [2867; 3120] | 3000 | 99 | [2827; 3170] |
| 45 | 3000 | 178 | [2767; 3269] | 3000 | 185 | [2723; 3317] |
| 60 | 2999 | 154 | [2833; 3292] | 3000 | 165 | [2767; 3338] |
| 90 | 3000 | 350 | [2534; 3646] | 3000 | 357 | [2456; 3672] |
| 120 | 3000 | 494 | [2500; 3993] | 3000 | 501 | [2388; 3999] |
| 180 | 3000 | 721 | [2427; 4621] | 3000 | 728 | [2277; 4606] |
| 240 | 2999 | 904 | [2399; 5205] | 3000 | 911 | [2208; 5192] |
| 360 | 3000 | 1082 | [2360; 5533] | 3000 | 1092 | [2143; 5569] |

Table 3 : standard uncertainties in $V_e(T)$ for two cases : i) with discretisation only (columns A to C), ii) with both discretisation and random errors (columns D to F)

# 4. REFERENCES

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# 5. APPENDIX

## 5.1 ESTIMATION OF STANDARD UNCERTAINTIES IN *R*, *h* AND *U*

Note: the calculations presented in this section are only given as additional information in order to illustrate how various approaches can be used in uncertainty assessment. Other and/or improved approaches can also be applied.

### 5.1.1 Uncertainty in *R*

As it is easier in practice to measure the pipe diameter, the pipe radius $R$ has been calculated from $n = 4$ measurements of the diameter $D$ carried out at various positions in the pipe cross section, as given in Table 4.

| $D$ (mm) |
|---|
| 1002 |
| 1000 |
| 997 |
| 1002 |

Table 4 : four measurements of the pipe diameter.

The mean value $\overline{D} \approx 1000.25$ mm and thus the mean radius $R = \overline{D}/2 = 500.125 \approx 500$ mm. The standard deviation $s$ of the four values of $D$ is equal to 2.3629 mm. The 95 % confidence interval (with $\alpha = 0.05$) for the mean value $\overline{D}$ is given by

$$\overline{D} - t_{1-\alpha/2}(\nu)\frac{s}{\sqrt{n}} \le \overline{D} \le \overline{D} + t_{1-\alpha/2}(\nu)\frac{s}{\sqrt{n}} \quad \text{eq. 24}$$

where $t$ is the Student value for $\nu = n-1 = 3$ degrees of freedom: $t = 3.1824$.

Assuming that the above 95 % confidence interval can be re-written

$$\overline{D} - 2u(\overline{D}) \le \overline{D} \le \overline{D} + 2u(\overline{D}) \quad \text{eq. 25}$$

the standard uncertainty $u(D)$ is determined by

$$u(\overline{D}) = \frac{1}{2}\left(t_{1-\alpha/2}(\nu)\frac{s}{\sqrt{n}}\right) \approx \frac{1}{2}\left(3.1824\frac{2.3629}{\sqrt{4}}\right) = 1.8799 \text{ mm} \approx 2 \text{ mm.} \qquad \text{eq. 26}$$

Then $u(R) = u(\overline{D})/2 = 0.9399$ mm ≈ 1 mm.

For calculations in this paper, one uses $R$ = 0.5 m and $u(R)$ = 0.001 m.

### 5.1.2 Uncertainty in *h*

The water level *h* has been measured with a 0-2 m piezoresistive sensor. The sensor was previously calibrated in the laboratory. Details are presented in Bertrand-Krajewski and Muste (2007): only key results are given in this appendix. Reference water levels have been measured in a Perspex column (height 3.5 m, diameter 0.2 m) with a class II certified 4 m long metallic meter, with a standard uncertainty less than 0.5 mm. For each reference water level, considered as a standard value, 12 repeated measurements have been made with the piezoresistive sensor. The data are given in Table 5.

| $x_i$ | $y_{i1}$ | $y_{i2}$ | $y_{i3}$ | $y_{i4}$ | $y_{i5}$ | $y_{i6}$ | $y_{i7}$ | $y_{i8}$ | $y_{i9}$ | $y_{i10}$ | $y_{i11}$ | $y_{i12}$ | $y_{i\,mean}$ | $s_i$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 399 | 399 | 400 | 400 | 400 | 400 | 400 | 400 | 399 | 399 | 399 | 399 | 399 | 399.50 | 0.5222 |
| 799 | 800 | 800 | 800 | 800 | 800 | 800 | 800 | 800 | 800 | 800 | 800 | 800 | 800.00 | 0.0000 |
| 1200 | 1201 | 1201 | 1202 | 1202 | 1202 | 1202 | 1201 | 1201 | 1201 | 1201 | 1201 | 1201 | 1201.33 | 0.4924 |
| 1600 | 1601 | 1601 | 1601 | 1601 | 1601 | 1602 | 1600 | 1600 | 1600 | 1600 | 1600 | 1600 | 1600.58 | 0.6686 |
| 2000 | 2002 | 2002 | 2002 | 2002 | 2002 | 2002 | 2001 | 2001 | 2001 | 2001 | 2001 | 2001 | 2001.50 | 0.5222 |

Table 5 : piezoresistive sensor calibration data. $x_i$ = reference value, $y_{ik}$ = repeated measurements with $k$ = 1:12, $y_{i\,mean}$ and $s_i$ = respectively mean value and standard deviation of the 12 $y_{ik}$ values. All values are in mm.

Ordinary least squares regressions have been made and compared by means of F-tests. The resulting optimal calibration function, with a residual variance $s_l^2 = 0.344398$, is

$$y = a + bx = 0.508854 + 1.000395\,x \qquad \text{eq. 27}$$

It is then possible to transform any measured value $y_0$ into the corresponding most likely true value of the water level $x_0$, and also to evaluate its standard uncertainty $u(x_0)$. Consider one single measured value $y_0$ = 701 mm. The most likely true value $x_0$ is calculated with eq. 28:

$$x_0 = \frac{y_0 - a}{b} = \frac{701 - 0.508854}{1.000395} \approx 700.2 \text{ mm} \qquad \text{eq. 28}$$

The standard deviation $s(x_0)$ is due to two independent contributions: i) the uncertainty in the measured value $y_0$, and ii) the uncertainty in the calibration curve expressed by the uncertainties in both coefficients $s(a)$ and $s(b)$. $s(x_0)$ is calculated by:

$$s(x_0)^2 = \frac{s_l^2}{b^2}\left(1 + \frac{1}{N} + \frac{(x_0 - \overline{x})^2}{\sum_i n_i (x_i - \overline{x})^2}\right) = \frac{0.344397}{(1.000395)^2}\left(\frac{61}{60} + \frac{(700.2 - 1199.6)^2}{19228814.4}\right) = 0.3543 \qquad \text{eq. 29}$$

Accordingly, adopting $u(x_0) = s(x_0)$ produces $u(x_0) \approx 0.6$ mm. The 95 % confidence interval (enlargement factor equal to 2) for $x_0$ is then given by $[x_0-2u(x_0), x_0+2u(x_0)] \approx [699.0, 700.4]$. The final result is expressed $x_0$ = 700.2 ± 1.2 mm.

The above result means that the sensor standard uncertainty, for $y$ = 700 mm, is equal to 0.6 mm, under stable laboratory or calibration conditions.

However, in the field (e.g. in a real sewer), the *in situ* measurement standard uncertainty is greater than 0.6 mm because the water level is not perfectly flat and stable but uneven with at least small waves, the free surface is not strictly horizontal due to turbulence and secondary flows, the exact position of the sensor in the pipe section is not known with a very high precision, etc. An *empirical global estimate* of this additional sources of uncertainty, *based on visual observations* made in various measurement locations in sewers, is evaluated to be about $u_r$ = 5 mm (see e.g. Bertrand-Krajewski *et al*., 2000, p. 223).

As a consequence, the *in situ* measurement standard uncertainty $u(h)$, with the above sensor, is assumed to be equal to

$$u(h) = \sqrt{u(x0)^2 + {u_r}^2} = \sqrt{0.6^2 + 5^2} = \sqrt{0.36 + 25} \approx 5\,\text{mm} \qquad \text{eq. 30}$$

For calculations in this paper, one uses $h$ = 0.7 m and $u(h)$ = 0.005 m.

During storm events and high flows, based on *in situ visual* observations, $u(h)$ may be higher, up to 15 mm. This indicates clearly that, if the water level sensor is of good quality with low uncertainties, the main source of uncertainty for *in situ* measurement is due to the natural turbulence and instabilities of water surfaces.

If detailed information is available, it could be possible to propose a simple empirical function to estimate $u(h)$ as a function of $h$ or of the discharge $Q$.

### 5.1.3 Uncertainty in *U*

The mean flow velocity $U$ is calculated from the information provided by the Doppler sensor. The Doppler sensor, when located on the pipe invert, is determining the mean velocity $V$ (m/s) within a conical volume characterised by three parameters: its angle of inclination related to the horizontal line, its angle of aperture, and its length. The two first parameters are given in manufacturer's specifications. The length, which determines the volume explored by the sensor, is not fixed and depends on factors like e.g. the variable suspended solids concentration or the ultrasound emission frequency. As the sensor explores the conical volume which is only a fraction of the cross-section, and as velocity profiles are not uniform through the cross section, the mean velocity $U$ (m/s) through the cross section is usually calculated by $U = kV$, where $k$ is a correction factor to account for geometry, flow regime, velocity profiles, etc.

Theoretically, $u(U)$ could be evaluated from $u(V)$ and $u(k)$ and by using the law of propagation of uncertainties. In practice, this is much more complex, as $u(V)$ and $u(k)$ are not known and difficult to assess. On the one hand, $u(V)$ is not constant and depends on the discharge conditions (pipe geometry, flow regime, depth of flow, etc.). Manufacturers may provide some values for $u(V)$, but they are only indicative and do not necessarily correspond to real conditions of use in sewers. Indeed, in most cases, they are estimated either i) by moving the sensor with a known velocity over a basin with still water, like for calibration of propeller velocity meters, which does not correspond to realistic vertical velocity profiles (see Bertrand-Krajewski *et al*., 2000, p. 371), or ii) by doing measurements under controlled laboratory conditions for limited circular pipe conditions with clean water, which cannot be transposed to real sewers. On the other hand, $k$ also changes with flow conditions (geometry, depth of flow, etc.) and estimating its uncertainty means than one has another and more precise method to estimate the discharge. These difficulties led us to use a more conservative and empirical approach.

A first attempt consisted to compare, in a real sewer, the mean velocity along a vertical profile measured with a calibrated propeller velocity meter and the mean velocity $V$ given by a floating Doppler sensor oriented toward the pipe invert (Bertrand-Krajewski *et al*., 2000, p. 374). Such experiments have been repeated later on (unpublished results) with a calibrated punctual electromagnetic velocity meter OTT Nautilus C2000. As these measurements are time consuming and cannot be easily carried out and repeated under various flow conditions in sewers, we concluded that, as an empirical conservative first approximation, $u(U)$ is around 0.05 m/s, for some circular or egg-shape man-entry sewers, and only for water levels between 0.15 and 0.4-0.5 m (for security reasons, it is not possible to carry out measurements in sewers with higher water levels occurring during storm events).

Obtaining more accurate *in situ* estimations of $u(U)$ requires independent, reliable and precise discharge measurements. More recently, we applied both salt and Rhodamine WT repeated tracing experiments, with one second time step continuous measurements of electric conductivity and fluorescence respectively, with calibrated sensors. These experiments may provide measurements of the discharge $Q$ with an enlarged uncertainty of less 6 % (preliminary results are given in Lepot *et al*., 2010). Knowing $Q$ and the water level $h$ also measured during the tracing experiments, one can then derive $U$ and its uncertainty. Repeated tracer experiments for various water levels and flow conditions will allow better estimating of $u(U)$ than the previous comparisons with velocity meters.

## 5.2 MATLAB SOURCE CODE FOR MC CALCULATIONS

```
R = 0.5 + 0.001*randn(1e6,1);
h = 0.7 + 0.005*randn(1e6,1);
U = 0.8 + 0.05*randn(1e6,1);
Q = R.^2.*(acos(1-h./R)-(1-h./R).*sin(acos(1-h./R))).*U;
result = IC95min(Q)
```

with IC95min the following function

```
function y = IC95min(V)
% this function gives a horizontal vector y = [m p1 p2]
% with m the mean value and p1 and p2 the boundaries of the shortest
% 95% coverage interval of the vector V
j = (0:0.01:5)';
m = prctile(V,j+95)-prctile(V,j);
[~, b]=min(m);
y = [mean(V) prctile(V,j(b)) prctile(V,j(b)+95)];
```